Mathematics Standards Study Group
Sample Problems

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Abstract

In July 2004 the Park City Mathematics Institute, under the direction of Herbert Clemens, hosted two workshops on states' K-12 mathematics standards. Both workshops were supported by the National Science Foundation. The first, July 21-24, was organized by Johnny Lott and was a meeting of the Association of State Supervisors of Mathematics, the National Council of Teachers of Mathematics, and some research mathematicians with an interest in K-12 mathematics education. The second workshop was the Mathematics Standards Study Group, July 25-28. This included twelve research mathematicians, many of whom had attended the first workshop, and was organized by Roger Howe.

The problem sets given here are part of the Proceedings of the Mathematics Standards Study Group. The objective is to illustrate how appropriate problems can support deeper learning of mathematics, and suggest how such problems can be incorporated into Standards Documents and the curriculum. The full Proceedings can be found on the PCMI web site http://pcmi.ias.edu .

Problem Sets: Introduction

We present five problem sets focussed on particular areas of K-12 mathematics. These sets are designed to reflect and expand on aspects of our recommendations for state standards. They do not attempt to provide a systematic account of the curriculum, but instead focus on particular issues that are among those we believe are most important. The sets are as follows.

- Set One concerns Problems for Pencil and Paper. In this set we give benchmarks for computational fluency, benchmarks that we believe are crucial if students are to be able to use mathematics effectively. A key recommendation of our group is that students develop the ability to compute fluently without use of a calculator. In our experience, students who cannot do numerical calculations without a calculator do not have an adequate base of knowledge and intuition for further studies of mathematics or science. This section focuses primarily on computations in arithmetic, but includes a few selected problems in algebra as well.

- Set Two concerns Patterns and Sequences. The study of patterns is widespread, providing a way to begin to develop concepts such as function. However, we are concerned that many of the problems in circulation, including ones on state tests, are actually misleading. We explain our concern and contrast harmful and helpful problems in this area.

- Set Three concerns Logical Reasoning in Mathematics. The ability to reason mathematically in a grade-appropriate fashion is of great value in understanding and using mathematics effectively. One aspect of such reasoning is contextual reasoning. Though such reasoning is certainly of importance, we recommend that the standards include material on mathematical reasoning that goes far beyond this. We describe what we have in mind in some detail and illustrate it with problems.

- Set Four concerns Data, Statistics, and Probability. These are mathematical topics of widespread utility. We offer problems on data and statistics, and on probability. We recommend that in state standards concerning data and its analysis, the emphases be on the representation and statistical description of data and on critical reasoning in the context of data analysis.

- Set Five concerns Lengthy Calculations. Here we describe calculations, too long for state tests, which nonetheless could be a valuable addition to the curriculum, reinforcing basic concepts and skills. Implicit in this section is our belief that classroom work should not be limited to material that will directly appear on state assessments.

1. Problems for Pencil and Paper

The foundation of mathematics learning is the mastery of whole number arithmetic and the place-value system. A similar mastery of fractions and decimals is a key goal for subsequent learning. An important aspect of this mastery is the ability to do arithmetic calculations accurately by hand, without a calculator. For computational fluency, students should be able to add and multiply single-digit numbers automatically and also do the corresponding subtraction and division problems very rapidly. Their ability to do these calculations—which are the foundation of all arithmetic computations—automatically and with complete accuracy should be developed fully in the early grades. They should also be able to multiply numbers by $10, 100$, etc. very rapidly. In addition, students should be able do the following operations correctly using only pencil and paper:

- add a set of multi-digit numbers, such as half a dozen 3-digit numbers
- subtract one multi-digit number from another
- mutliply two multi-digit numbers, such as a 4-digit number by a 2-digit number
- divide two multi-digit numbers, such as divide a 4-digit number by a 2-digit number
- combine understanding of place value with addition, subtraction, multiplication, and division (e.g. go from 36×12 to 36×120 and 36×1.2)
- translate between improper fractions and mixed numbers
- add, subtract, multiply, and divide fractions
- find the decimal equivalent of a given fraction and the fraction equivalent of a given decimal
- add, subtract, multiply, and divide decimals
- work with percentages, and convert fluently between percentages, decimals, and fractions
- add, subtract, multiply and divide numbers written in scientific notation, and convert between scientific notation and standard notation.

Indeed, as explained in the Lead Essay, we believe that a mastery of these pencil-and-paper computations is crucial to a solid understanding of numbers and arithmetic. This level of competency also enables students to understand the individual steps in complex computations. Such computational fluency is also useful in many real-world situations, from financial transactions to grocery shopping.

We emphasize that the ability to do arithmetic with fractions by hand is important. It is necessary for success in algebra; a skill with decimals but not fractions (for example, being able to handle operations with fractions solely by means of a calculator) is *not* adequate. Indeed, fluency in the arithmetic of fractions is crucial background for developing fluency in the algebra of fractions. For fractions appear in algebra in the form of algebraic expressions that cannot be treated as numbers on a calculator. Besides this, fractions play a key role in understanding statistics and probability (and conversion of fractions to decimals makes its almost impossible to trace an error or to easily assess how the answer will change if a problem is changed slightly).

In the following problems we illustrate these skills. We also include several problems in algebra and basic trigonometry of a similar flavor.

1:

$$234 + 582 = \qquad 901 - 74 =$$
$$582 \times 7 = \qquad 392 \div 7 =$$

Discussion: Prior to doing these calculations, students must be able to add and multiply one digit numbers *automatically* (that is, without pencil and paper, and quickly, without the need to compute) with complete accuracy. They should also be able to do the inverse problems (subtraction, division, such as $16 - 9$, $56 \div 8$) automatically. The ability to do these basic calculations automatically is fundamental for all that follows.

2:

$$\begin{array}{r} 357 \\ +\ 842 \\ +\ 2,392 \\ \hline \end{array}$$

3: Calculate 37×79. Then calculate $37 \times .79$.

4: Calculate

$$23\overline{)805} \qquad \text{and also} \qquad 2.3\overline{)8.05}$$

5: Convert each of the following to a whole number, a proper fraction in lowest terms or a mixed number with fractional part in lowest terms.

$$\frac{7}{4} \qquad 25\% \qquad 23.24 \qquad \frac{632}{27} \qquad .\overline{9}$$
$$62.5\% \qquad \frac{63}{784} \qquad 0.0412 \qquad 0.03125\% \qquad .3\overline{25}$$

Discussion: In the problem above, for convenience we have placed somewhat different types of number representations together. For students, including these ten calculations as parts of a single problem might significantly increase the difficulty.

6: Which of the following fractions equals 0.8: 3/4, 4/5, 5/4, 2/3, 3/2, 4/3 ?

7: Write 84 and 350 each as a product of prime numbers. Then find their greatest common divisor.

Discussion: Factoring a number into primes is a useful tool in finding greatest common divisors and least common multiples, and in determining when one number is a multiple of another.

8: Write the answers to the following problems as fractions, proper or improper, using the number one in the denominator in case of a whole number answer.

$$\frac{2}{3}+\frac{5}{6}= \qquad \frac{5}{6}-\frac{2}{3}= \qquad \frac{2}{3}\times\frac{5}{6}= \qquad \frac{5}{6}\div\frac{2}{3}=$$

$$\frac{2}{3} \text{ of } 12\frac{1}{2}\% = \qquad 3.225+1.725= \qquad 4.75\div(5/7)=$$

$$17\% \text{ of } 17 = \qquad \frac{5}{6}\times\left(\frac{3}{4}+\frac{1}{3}\right)= \qquad \left(\frac{5}{6}\times\frac{3}{4}\right)+\frac{1}{3}=$$

9: Write the answers to the following problems as fractions, proper or improper, with natural numbers (that is, positive integers) as denominators.

$$^{-}7+{}^{-}4= \qquad ^{-}7-{}^{-}4= \qquad ^{-}7\times{}^{-}4= \qquad ^{-}7\div{}^{-}4=$$

$$\frac{^{-}5}{3}+\frac{7}{2}= \qquad ^{-}3\tfrac{4}{5}\times 2\tfrac{1}{7}= \qquad ^{-}2.75\div{}^{-}0.25= \qquad 4-({}^{-}3-7)=$$

$$(4-{}^{-}3)-7= \qquad 4\div({}^{-}5\times 2)= \qquad (4\div{}^{-}5)\times 2= \qquad ^{-}4\times({}^{-}6-17)=$$

10: Express the answers to the following calculations as fractions, whether proper or improper.

$$\frac{13}{35}-\frac{5}{21}= \qquad \frac{^{-}19}{28}+\frac{47}{70}= \qquad \frac{22}{15}-\frac{^{-}4}{21}-\frac{6}{35}=$$

11: Write each of the following expressions as a fraction having denominator 36.

$$\frac{2/3}{3/2} \qquad \frac{5/2}{6} \qquad \frac{26}{72} \qquad \frac{14/(4/5)}{45}$$

Discussion: Placing fractions over a common denominator enables one to easily order them on the number line, and is also useful for some calculations.

12: Give your answers to the following calculations as either integers or as decimals with at most three significant figures.

$$5^4= \qquad ({}^{-}5)^4= \qquad 4^5= \qquad 3^{-5}= \qquad (7/5)^2= \qquad ({}^{-}1.7)^3=$$

13: Write each answer or approximate answer as a power of 10 times a number having one nonzero digit to the left of the decimal point and two digits to the right of the decimal point.

$$(3.71 \times 10^3) + (2.12 \times 10^2) = \qquad (5.20 \times 10^4)\times \; (2.30 \times 10^{-3}) =$$
$$(3.54 \times 10^0) \div (7.00 \times 10^{17}) = \qquad (3.1 \times 10^4)^3 =$$

14: Solve each equation for x.

$$\frac{6}{25} + \frac{6}{25} + \frac{6}{25} = \frac{x}{75} \qquad \frac{x}{2} = \frac{3x}{4} - 1 \qquad \frac{2}{x} + 7 = 12$$
$$.01 + .6x = .05 \qquad x^2 + 3x + 2 = 0.$$

15: Write each of the following as sums and/or differences of monomials.

$$(3x^2 - 2)(x + 5) = \qquad 7x[(3x - 2) + 5(x - 6)] = \qquad (2x^2 + 5x - 7)(x^2 + 6) =$$

Discussion: Asking students to simplify is sometimes ambiguous because, for instance, a factored form is simplest for many purposes.

16: Write the following expressions in factored form in such a way that in any factor in which x appears it only appears to the first power and its coefficient equals 1.

$$x^2 - 9 = \qquad 4x^2 - 36 = \qquad x^2 + 6x + 8 = \qquad 3x^2 + 12x + 12 =$$
$$\sqrt{5}\,x^2 - 9\sqrt{5} = \qquad x^2 - 13 = \qquad 2x^2 - 8y^2 = \qquad \sqrt{6}\,x^2 - \sqrt{96} =$$

17: Simplify each of the following expressions for those values of x for which the denominator does not equal 0.

$$\frac{x^2 - 9}{x - 3} \qquad \frac{x^2 - 9}{x + 3} \qquad \frac{x^2 + 5x + 6}{x + 3} \qquad \frac{3x^2 - 48}{6x + 24}$$

18:

$$\sin 30^\circ = \qquad \tan(\pi/4) = \qquad \cos 45^\circ = \qquad \arcsin(\sqrt{3}\,/2) =$$

Discussion: Some might prefer that these values of the trigonometric functions be memorized. We have put them here because someone who does not have them in memory can (and should be able to) do a short calculation to obtain the correct values.

The following problems are somewhat more technical, but illustrate skills which high school students should master in order to be adequately prepared for a college major in a discipline that makes use of quantitative methods. Since high school students who do not anticipate such a major (or even college attendance) may later change, it is advisable to encourage as many as possible to achieve this mastery.

19: Complete the square to write each of these expressions as $ap(x)^2 + b$ where $p(x)$ is a polynomial and a, b are numbers.

$$x^2 + 3x - 4 \qquad -2x^2 + 8x - 3 \qquad x^6 + 4x^3 + 7$$

Then sketch the graphs of these functions.

20:

$$\begin{aligned}(x^4 + x^3 - x - 1) \div (x^2 - 1) &= \\ [(x^2 - 1)/(x^2 + 1)] \div [(x + 1)/(x^2 + 1)] &= \\ \frac{-2a}{a^2 - 1} + \frac{1}{(a-1)^2} &= \end{aligned}$$

21: Rationalize the denominator and write the new numerator in a form not requiring parentheses. Then start over, rationalizing the numerator and writing the new denominator without parentheses.

$$\frac{\sqrt{x+4} + 9}{\sqrt{x+4} - 2} = \qquad \frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}} = \qquad \frac{x^{1/3} - 5}{x^{2/3} + 5x^{1/3} + 25} =$$

22: Find the coordinates of the point where the lines $2x + 3y = 4$ and $3x + 8y = 5$ intersect.

23:

$$\frac{1}{3}\log_b 64 = \qquad \log_3 81 = \qquad -\log_b(1/c) = \qquad \log_{10}(1/1000) =$$

24: Give simple equivalent expressions involving tan.

$$\cot(\tfrac{\pi}{2} - \theta) = \qquad \cot(\theta - \tfrac{\pi}{2}) = \qquad \frac{2\tan\theta}{1 - \tan^2\theta} = \qquad \tan(\pi + \theta) =$$

2. Patterns and Sequences

Mathematics is crucial in science and the social sciences because it allows one to express precisely the relationship between different quantities. The study of such relationships is the heart of algebra. It is natural to want to introduce children to mathematical notions such as variables, relations, and functions as soon as possible, and this is often done through the study of patterns. For example, the doubling rule beginning with the number 1 gives the sequence

$$1, 2, 4, 8, \ldots,$$

and in later mathematical studies this leads to a discussion of the function $y = 2^x$ and to the notion of exponential growth.

It is important in discussing and in testing patterns that the basic idea of mathematics as a rigorous subject involving disciplined reasoning not be lost. (See the Lead Essay, Principles for School Mathematics 3 and 4.) For example, if students are to investigate the sum of the first n odd numbers

$$\begin{aligned} 1 &= 1 \\ 1+3 &= 4 \\ 1+3+5 &= 9 \\ 1+3+5+7 &= 16 \end{aligned}$$

and notice that the answer is the perfect square n^2, it is important that they learn that this is a mathematical fact which is not established for all n simply by checking some cases, but which can be (and will be!) established at an appropriate later time in their studies. (See the discussion of this topic in the NCTM Standards for Algebra, Grades 3-5.)

The difference between noticing a pattern and showing conclusively that the observed pattern continues is often overlooked. To explain why this is not automatic, consider the sequence

$$1, 2, 4, ?.$$

One way to continue the sequence is to assign the value $? = 8$; this can be arrived at by the doubling rule, for example. However, if one simply asks to continue this pattern then in fact *any number can be assigned as the value of* ?. That is, there is no single mathematically valid answer here which can be arrived at by mathematical reasoning. To emphasize this, let us mention a few different ways of continuing the sequence which also arise from natural mathematical constructions:

(i) $124/999 = .\overline{124} = .124124124\ldots$

(ii) $138/1111 = .\overline{1242} = .124212421242\ldots.$

(iii) The next term is $? = 7$. This arises from a geometric problem: count the number of regions into which lines in "general position" divide the plane. For instance, if there were no lines, there would be 1 region—namely, the entire plane; if there were one line, the two sides of that line would make 2 regions; two intersecting lines would divide the plane into 4 regions (here "general position" means not parallel). The next term (the number of regions obtained from three non-concurrent pairwise intersecting lines) is 7.

The point here is that though context may allow one to select a particular next term which is most likely to fit a given collection of data or to apply to a given specific situation, it is simply incorrect to say that there is a single next answer to the problem in the absence of additional information.

This leads us to a crucial principle:

Test and homework problems involving patterns and sequences should not rely on unstated assumptions, nor should they imply that there is a unique answer when this is not mathematically justified. In particular, a given sequence of numbers with no other information stated has infinitely many continuations.

Adhering to this principle is crucial if students are to be fairly tested, are to develop proper notions of mathematical reasoning, and if they are to find mathematics presented as a series of logical steps rather than rules without justification.

In fact, claiming that an observed pattern generalizes without justification not only misleads students about the role of mathematical reasoning in understanding and using mathematics, it can also lead to false conclusions. Students who look at the first five terms of the sequences $(n+2)^2$ and 2^n (namely $9, 16, 25, 36, 49$ and $2, 4, 8, 16, 32$, respectively) might expect that the term in a given position of first sequence is always larger than the term in the corresponding position of the second sequence, but this is not true from $n = 6$ onwards. If students are asked to consider a circle with several points selected on its circumference and draw all the chords with these as endpoints, then they will notice that these chords divide the circle into a number of regions. For small numbers of points $n = 1, 2, 3, 4, 5$ they can observe that the number of regions is $1, 2, 4, 8, 16$ (respectively) and they may think that the number is always 2^{n-1}. But this is not true; the number of regions when $n = 6$ is 31 and not 32. Similarly in real-world applications, extrapolation from data is important but context is crucial; if the share price of a certain stock at the end of March, April, and May is \$10, \$20, and \$30 (respectively), it would be rash to expect the price to be \$40 at the end of June simply on the basis of this information.

With this as background we illustrate problems that meet and do not meet the basic criterion above. We label the latter harmful since they in fact deliver a false impression of mathematics itself.

Harmful problems:

1: What is the next term in the sequence $1, 2, 4$?

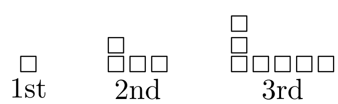
2: Give the continuation of the pattern which begins $3, 1, 4, 1, 5$.

Discussion: The objections above apply. In fact, as noted above, there are infinitely many answers, and with the information given there is no mathematical reason to prefer any one of them.

3: The Fair Meadows County Library charges a fine of $0.50 for a book that is two days late and $0.75 if it is three days late. What is the late fee for a book that is returned one day late? Two weeks late? One hundred days late?

Discussion: This problem cannot be solved with the information given. What if the library has a one-day grace period? What if it has a system of fines that is not linear, such as a cut-off at the cost of the book plus a service charge? The point here is that someone working on this problem is required to make an assumption that is not stated in the problem.

4: Each arrangement in the pattern below is made up of tiles. How many tiles are there in the fifth arrangement in this pattern?



Discussion: Once again the "pattern" is not described mathematically. There is no way to answer the question without guessing the particular continuation that its author had in mind. In fact there are infinitely many other continuations, each of which is consistent with the information given in the problem. The question could be changed to address this problem by stating that the successive arrangements are obtained by adding boxes to the top of the leftmost edge and to the right of the bottom edge and nowhere else and that the numbers of tiles on these edges are described by arithmetic progressions.

We also discuss two other problems that would satisfy, albeit minimally, the criterion stated above, but which do not seem of great value to us:

5: State a rule which gives a sequence beginning $3, 1, 4, 1, 5$, and use your rule to find the next term in the sequence.

Discussion: Though well-intentioned, this is somewhat misleading in that the rule which states by caveat that the sequence is 3,1,4,1,5,1000, continuing and repeating forever is mathematically just as valid as any other rule (even one more ostensibly natural such as the digits of π). So there is almost no mathematical content in the question.

6: (For elementary school students) Find 5 patterns among the entries of Pascal's triangle:

$$\begin{array}{c} 1 \\ 1\ \ 1 \\ 1\ \ 2\ \ 1 \\ 1\ \ 3\ \ 3\ \ 1 \\ \vdots \end{array}$$

Discussion: This problem makes no hidden assumptions. The issue is that there is no great gain in mathematical knowledge for elementary students who may notice that entries of the rows add up to powers of 2, etc. Prior to work done counting combinations, such investigations do little to contribute either to the mastery of core topics or to the development of rich mathematical reasoning skills, and seem to us of limited value.

Helpful problems:

7: Which of the following rules gives rise to a sequence beginning $1, 2, 4$? For each of the rules that does so, find the next term predicted by that rule. (7-a) the doubling rule: the next term is twice the preceding one. (7-b) add one to the first number to get the second, two to the second number to get the third, and in general add N to the N-th number to get the $(N+1)$-st. (7-c) triple any number in the sequence and subtract 1 from this to get the next number. (7-d) add 1 to the numbers in odd positions in the list and add 2 to the numbers in even positions in the list in order to get the next. (7-e) square any number in the sequence and add 1 to the result to get the next number.

8: Fill in the next three terms of the following sequence in which the rule for all but the first two terms is that each term is 1 more than the sum of the preceding two terms: $1, 2, 4, \_\_, \_\_, \_\_$.

Discussion: Note that the difficulty in understanding the rule from prose is nontrivial and this is the most difficult aspect of this problem. But this is the kind of reading that is very important in mathematics.

9: Notebooks are sold at a fixed price per notebook. At a special sale, if a customer purchases from 2 up to 6 notebooks, the fixed price is reduced by 5 cents times the number of notebooks purchased. If 3 notebooks cost \$2.25, how much would 1 notebook cost? How many notebooks would you need to purchase to save 1/3 of the original price on each notebook?

10: The Fair Meadows County Library charges a fine of a fixed amount per day for each day a book is overdue. If the fine is \$0.50 for a book that is two days late and \$0.75 if it is three days late. what is the late fee for a book that is returned one day late? Two weeks late? One hundred days late?

We note that a more systematic study of sequences may be undertaken using algebraic symbolism. This leads to the definition and discussion of arithmetic and geometric progressions, to the notion of recursion which is tied to computer science, and to proof by mathematical induction. The algebraic symbolism used in recursions makes it easier to write questions for advanced middle school or high school students than for K-6 students. Here are five such questions with different levels of difficulty.

11:

(a) What are the first five terms of the arithmetic progression which begins 2, 6? What is the 20th term? (For advanced high school students: What is the sum of the first 20 terms?)

(b) What are the first five terms of the geometric progression which begins 2, 6? What is the 20th term? (For advanced high school students: What is the sum of the first 20 terms?)

12: Suppose that $a_1 = 1$, and for $n > 1$, that $a_n = 2a_{n-1}$. So,

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 2a_1 = 2 \times 1 = 2 \\ a_3 &= 2a_2 = 2 \times 2 = 4\,. \end{aligned}$$

Find a_4, a_7, and a_{14}.

13: It is given that $a_n = 2a_{n-1}$ and that $a_6 = 12$. Find a_1, a_3, and a_9. Then find a closed (i.e. explicit) formula for a_n for arbitrary n, and prove your formula.

Discussion: The preferred solution for the first part is to divide 12 by 2^{6-1}, but it could also be solved by successively dividing by 2 five times. This part also reinforces the arithmetic fluency called for in many state grade 3-7 standards.

14: It is given that $b_1 = 1$ and that for $m \geq 1$, that

$$b_{m+1} = m + b_m\,.$$

Find $b_7 - b_5$.

Discussion: This problem is related to the geometric problem described in part (iii) of the discussion of sequences which begin $1, 2, 4$.

15: A sequence begins with $c_0 = 1$, $c_1 = 2$, and continues for $m \geq 0$ via the recursion

$$c_{m+2} = 1 + c_{m+1} + c_m\,.$$

Find c_6.

Discussion: It happens that the sequence in this problem and the one in problem (13) both begin as $1, 2, 4, 7$. But the next term in the preceding problem is 11 while the next term here is

12. Once again, we see an illustration of the principle that there is no unique way to continue a finite sequence of numbers in the absence of additional information.

As a concluding remark, we note that patterns do have a place in mathematics, *when combined with mathematical reasoning*. Since it may be easy to detect a pattern and there is a temptation to let that be a substitute for careful mathematical reasoning, it is important that teachers present the detection of possible patterns as only the first step in a more detailed investigation, *and not be satisfied with that step alone*. As an example, one could ask students to investigate whether or not $n^2 - n + 5$ is prime as n ranges over the whole numbers. Though this expression is prime for $n = 0, 1, 2, 3, 4$, in fact, it is composite for $n = 5$, and also for $n = 10, 15, 20$. Students who notice this could then *prove* that it is composite for all integers n that are mutliples of 5 except $n = 0$. Noticing that the expression is also composite for $n = 6, 11, 16, 21$ leads to another provable fact: values of n that are one more than a multiple of 5 yield composite numbers with the single exception that $n = 1$ yields the prime number 5. (The proof involves rewriting the polynomial as $n(n-1) + 5$.) The process of detecting patterns *and then using careful mathematical reasoning to establish them* is a useful technique in mathematical knowledge-building.

3. Logical Reasoning in Mathematics

Many state standards emphasize the importance of reasoning. We agree—disciplined reasoning is crucial to understanding and to properly using mathematics. Before presenting problems that require such reasoning we offer a few remarks. First, though 'reasoning' is widely mentioned in state standards, it is apparent that this word is used to describe many different things. For example, students who are asked

> *You want to purchase a bookcase to hold 37 books. If each shelf can hold 7 books, how many shelves should the bookcase you purchase have?*

need to compute that 37 divided by 7 is 5 with a remainder of 2, and reason from this that to hold all the books one needs 6 shelves (and not just 5!). In almost all applications of mathematics the formal procedures (in this case division with remainder) must be supplemented by an understanding of the context and by reasoning about this context. This type of contextual or situational reasoning is certainly important, and questions such as the one above have a solid place in the curriculum and on state tests. However, we wish to highlight here that *the ability to use and to understand mathematics requires logical reasoning that goes far beyond this.* By mathematical reasoning or logical reasoning we mean—and we believe that state standards should include in a significant way—precise deductive reasoning.

Deductive reasoning skills are crucial in mathematics (as well as in many other walks of life). However, achieving a high level of student competence in deductive reasoning will require great care on the part of teachers. Indeed, the level of precision necessary to communicate mathematics is quite high, and it is easy for the generalist to overlook this. 'Three times higher than' does not mean the same thing as 'three times as high as' . The phrases 'six divided by three', 'six divides three', and 'six divided into three' all have different mathematical meanings. It is difficult to use deductive reasoning effectively without attention to this precision. All involved should be encouraged to cultivate careful reading, listening, presenting, and writing skills in the domain of mathematics.

The problems below all involve deductive reasoning, but they are of several types. First there are problems that require an understanding of quantifiers ('for some', 'for all'), negation ('not all', 'some are not', etc.), and other mathematical language. Students need to understand this language and to evaluate whether a given statement involving quantifiers is true. They also need to learn that a statement that is only true sometimes is mathematically false (e.g. the statement 'the product of two irrationals is irrational' is false).

Second are problems that require understanding of 'if-then' deductive reasoning. Such problems come in a wide variety of levels and a wide range of sophistication, including multi-step implications. It is useful for working with 'if-then' statements to know that a statement and its contrapositive are logically equivalent and to be able to formulate the contrapositive of a given statement. However, this is only one limited part of working with 'if-then' statements. It is the ability to carry out multi-step deductive reasoning and the ability to detect incorrect implications and explain why they are wrong that are the primary goals here. At the highest level are problems that couple this reasoning with algebra or geometry.

Problems involving quantifiers and other mathematical language

1: Decide which of the following statements are true and which are false. Explain your answer.

(a) Some whole numbers are integers.

(b) Some whole numbers are not integers.

(c) Some integers are whole numbers.

(d) Some integers are not whole numbers.

(e) Not all integers are whole numbers.

(f) All integers are whole numbers.

(g) Different fractions necessarily represent different rational numbers.

(h) Between any two rational numbers there is at least one other rational number.

(i) A system of two simultaneous linear homogeneous equations in two unknowns necessarily has at least one solution.

Discussion: Statements (a)–(f) involve three things: understanding the quantifiers 'some' and 'all', understanding that 'not' is attached to the noun that follows it (as opposed to negating the entire sentence), and understanding the concepts 'integer' and 'whole number'. The fact that 'not' sometimes negates an entire sentence and other times negates only an aspect of the sentence may cause some students difficulty (a difficulty that also occurs in ordinary communication). However, with explanations from the teacher, students can become proficient at analyzing and explaining such statements in the early grades.

Statement (g) requires understanding the phrase 'necessarily represent' and understanding the difference between a statement sometimes being true and always being true. The mathematical fact involved is one that must be mastered in order to work with fractions.

The problem of deciding whether statement (h) is true or false and explaining the reason is a somewhat sophisticated problem about a basic concept. The logic aspect of the problem is centered around the use of the phrase 'at least' (a phrase that is especially difficult for those who are just learning English) and around the requirement that the answer be explained.

The last statement involves standard mathematics terminology, some mathematical knowledge, and a correct understanding of the phrase 'at least'.

2: Decide which of the following statements are true and which are false. Explain your answer.

(a) All equilateral triangles are isosceles.

(b) Some isosceles triangles are right triangles.

(c) Some right triangles are isosceles.

(d) Some equilateral triangles are right triangles.

(e) Some isosceles triangles are right triangles and equilateral.

(f) No isosceles triangle is equilateral.

(g) Not all isosceles triangles are equilateral or right triangles.

Discussion: These are similar to problem 1, but based on geometry, in this case on the definition of various kinds of triangles. For part (g), note that in mathematics 'or' means 'one or the other or both'.

Problems 1 and 2 are meant to be representative; other variations which concern the same aspect of mathematical reasoning in the context of other aspects of math knowledge are readily constructed.

Problems involving 'if-then' statements and deductive reasoning

3: Which of the following statements is logically equivalent to "If it is Saturday, then I am not in school."?

(1) If I am not in school, then it is Saturday.

(2) If it is not Saturday, then I am not in school.

(3) If I am in school, then it is not Saturday.

(4) If it is Saturday, then I am in school.

Discussion: This is Problem 8 on Regents High School Examination Mathematics A of the State of New York for June 17, 2003. It requires a student to know what ‘logically equivalent’ means (a high school student should know the meaning of this phrase). The student must also be able to work with ‘if’, ‘then’, and ‘not’. A nice feature of this problem is that it is written in the present tense, thus avoiding any concerns about causality.

One might contrast this with the problem “What is the inverse of the statement: ‘If John cuts the grass, then he earns $8.00’ ?” This tests only the definition of the ‘inverse’ of an ‘if-then’ statement, a definition which is considerably less important than the contrapositive (or the converse). It does not test deductive reasoning at a high level.

4: Which of the following statements has a true converse?:

(a) If two triangles are congruent, then they have the same perimeter.

(b) If two squares are congruent, then they have the same area.

(c) If two rectangles have the same perimeter, then they have the same area.

(d) If the area of a triangle is less than 1, then each of its sides has length less than 1.

Discussion: These problems illustrate that a statement and its converse may both be true, both be false, or one may be true and the other false. It is a common mistake to confuse the truth of a statement with the truth of its converse, both in mathematics and in the larger world. The ability to detect such potentially false reasoning is a basic life skill.

5: Some kings have beards. All kings wear red. All men who wear red are tall. Based solely on this information, which of the following statements are true? (a) Some men with beards are tall. (b) All tall men are kings. (c) Not all small men with beards wearing red are kings.

Discussion: This is a familiar type of problem and we do not list the possibilities exhaustively. Such problems should be mastered.

6: A,B,C,D, and E are friends, no two of whom have the same height. B is taller than C, A is not the tallest, E is taller than B, D is taller than exactly two of the others, and D is taller than B. What is the order of the heights of these friends from tallest to shortest?

Discussion: Once again, this is a familiar type of deductive reasoning problem. The point is that there is one answer and it can be arrived at deductively.

7: Joshua and Joel together have the same number of scarves as Sarah and Samantha together. Also, Joshua has more scarves than Sarah. What can you conclude?

Discussion: This problem is suitable for the elementary level, but it may be more useful for a classroom discussion or homework than for a test. One can also ask the related problem in an algebraic format: If $a + b = c + d$ and $a < c$, what can you conclude? Similarly, some of the algebraic statements in problem 8 below can be made into word problems.

8: Decide which of the following statements about real numbers are true and which are false. If the statements are true, give an argument which shows this. If the statements are false, give a counterexample.

(a) If $a < b$ and $c < d$ then $ac < bd$.

(b) If $a^2 < b^2$ then $a < b$.

(c) If $a^3 < b^3$ then $a < b$.

(d) If $(x + y)^2 = x^2 + y^2$ then $x = 0$ or $y = 0$.

(e) If $x^2 = y^2$ then $x = y$.

(f) $x^3 = y^3$ if and only if $x = y$.

Discussion: Statement (a) is a good example of an erroneous implication one might write in an algebra class. Plausible-looking is not the same thing as true! Statement (c) requires understanding a property of the function $f(x) = x^3$. Statement (d) is true and the student should be able to give a complete proof by the time they complete a first course concerning algebra. Statement (e) is the converse of a true statement. However, it is false. Just as in the geometric context of problem 4, it is important for algebra that students understand clearly that the converse of a true if-then statement need not be true. Statement (f) uses the phrase 'if and only if'.

9: Malcom says that

$$\frac{8}{11} > \frac{7}{10}$$

because $8 > 7$ and $11 > 10$. Even though it is true that $\frac{8}{11} > \frac{7}{10}$, is Malcom's reasoning correct? If Malcolm's reasoning is correct, explain why clearly; if Malcolm's reasoning is not correct, give Malcolm an example that shows why not. [This problem comes directly from *Mathematics for Elementary Teachers* by S. Beckmann, Pearson Addison Wesley 2005, pg. 93, problem 15.]

Discussion: In this case the statement $8/11 > 7/10$ is true but the reasoning offered to justify it is not correct. Indeed, the justification offered is based on the false statement: 'If $0 < a < b$ and $0 < c < d$, then $\frac{a}{c} < \frac{b}{d}$.' It is important that students learn that such reasoning, even when used to justify something true, is not mathematically valid. Note that though the algebraic form of the false statement requires sophisticated algebra skills to understand (and debunk), the problem as stated is suitable for middle school students.

In conclusion, we emphasize that the mathematical reasoning skills whose development is promoted here should also play a role in the way that mathematics is presented in the curriculum. To illustrate this, we offer two problems that could form bases for classroom discussions that involve mathematical reasoning.

10: Your best friend knows that ‘two wrongs don’t make a right’. This friend can not believe that for numbers, a negative times a negative is a positive. Explain to your friend why the product of two negative numbers must always be positive.

Discussion: This basic fact about multiplication is sometimes presented to students as a “rule” without full justification, but it can be explained. The explanation requires an application of the distributive law and is somewhat subtle.

11: If the sum of the digits of an integer, written in base 10, is divisible by 3, then the number itself is divisible by 3. Explain why this is true for integers with two or three digits.

Discussion: This is a sophisticated problem concerning the reason why an ‘if-then’ statement is true. The solution requires an understanding of the base 10 number system and the knowledge that 10 and 100 each have remainder 1 upon division by 3. Once students have understood this divisibility test for two and three digit numbers one could ask them to explain it in general.

4. Data, Statistics, and Probability

Data, statistics, and probability are mathematical topics of widespread utility. In this set of problems, we treat these subjects.

There are several separate, albeit related, topics of concern here. The first is data and the various methods of representing it. Representation of data may be graphical (bar charts, scatter plots, and the like) or by means of statistics that describe the data (mean, median, standard deviation, quartiles, etc.). It is important that students learn the most common of these methods and, for statistics, the calculations that give the statistics. Many state standards give the impression that good representation of data is rather straightforward and that the focus should be on interpretation. Actually, deciding on an appropriate representation itself requires good judgment, often in combination with arithmetical, algebraic, and/or geometrical understanding. *We believe that in state standards, one emphasis as regards Data and Analysis should be on the representation and statistical description of data*; see problems 1-5 below.

Another part of statistics treats the topic of drawing conclusions and making suitable predictions based on data. *We believe that the standards of some states as regards data analysis and experimental design should be significantly changed.* Instead of an emphasis on *making predictions*, we believe that the focus should be on *critical reasoning* in the context of data analysis. In particular, we recommend that the curriculum in this area include increased contrasting of valid and invalid arguments involving statistics. In the problems below, we give some examples.

A first reason for this recommendation is that it lays the foundation for the use of statistics in other courses, where subject-specific knowledge is often important, if not crucial, to a full understanding of the data and its implications. Indeed, we embrace including additional work with statistics in the science and social sciences curriculum. (We do, however, believe that the actual carrying out of all but the simplest experiments belongs in such classes rather than in the mathematics classroom.) Second, it is easy to misapply statistical methods (e.g. by treating dependent data as if it were independent), resulting in conclusions that are seriously flawed for non-obvious reasons. We believe that it is important for citizens to have a good understanding of such issues in order to critically analyze arguments using statistics. Third, in general we have great concern that students not be told in essence to plug in formulas that they may not genuinely understand and then to trust the answers. In the context of statistics we believe that this concern can be well-addressed by an increased emphasis on critical reasoning.

Let us also note that state examination questions in data analysis and in the related area of experimental design all too often, shockingly, reflect a real misunderstanding of this subject. Students are asked to draw a line of best fit when there is no indication that the problem is linear or of what "best fit" means, to design experiments without the proper mathematical background in this area (e.g. a discussion of correlation), to model data without being able to measure whether the model fits the data well, and to draw conclusions from experiments though those conclusions cannot be justified without additional, unstated, hypotheses. These problems suggest a certain overreaching. We believe that students would be far better served by learning instead of mislearning. It may be wiser to cover less material about statistics but to cover it more thoroughly, and then to encourage students to continue to learn about statistics as their mathematical background increases. We also believe that, outside of descriptive statistics, this material is best taught in a focused way once students already have a solid mathematical foundation in arithmetic and middle-school mathematics (including algebra), rather than being presented in smaller, continuing, doses.

Problems related to probability are also given in this section, primarily to highlight the distinction between probability, which is a purely mathematical subject motivated by real-world connections, and concerns with data and statistics in the real world. We explain this relation further in the subsection on Probability below.

Data and Statistics

1: Make a bar chart showing the population in millions of the 10 provinces and 3 territories of Canada (counting Northwest Territories as a single territory).

Discussion: Typically such a problem would be accompanied by a table showing the population in each province and territory (omitted here). This problem, an exercise in presenting data nicely for the reader, is consistent with the standards of many states.

Bar charts are occasionally called bar graphs.

2: Convert the bar chart of Problem 1 to a bar chart representing percentages of the total Canadian population.

Discussion: This problem is for students who have learned about the connection between decimals and percentages. The bar chart is the same as for Problem 1 except for labels.

3: Construct a bar chart showing the average populations per square mile of the provinces and territories of Canada. Make sure that the bar chart fits on one piece of paper but is not minuscule.

Discussion: It would be natural to use a calculator for this problem. Here, it would be a mistake to ask students to convert the bar chart to one representing percentages of the whole, since these data are themselves percentages that have been obtained by dividing by different numbers for different items.

Problems 1–3, Additional Discussion: There are many methods of presenting data. It is more important that students learn the advantages and disadvantages of the most common methods rather than obtain an encyclopedic knowledge of less common methods. Though we have not included problems about pie charts, such charts are popular in newspapers and other data presentations and should be discussed.

In the next two problems we are concerned with the statistics that can be calculated from numerical data as an aide to describing the data: average (also called mean), median, quartiles, quantiles, standard deviations, and variances. Students should understand these descriptors, and also know which changes in data affect the mean mostly and which have more influence on the median.

4: At our company, the mean annual salary is $100,000; the median annual salary is $25,000. Our company has 1,000 employees.

(a) Can the total annual amount allocated to paying salaries be determined from just the median salary and the number of employees? If it can, do so.

(b) Can the total annual amount allocated to paying salaries be determined from just the mean salary and the number of employees? If it can, do so.

Discussion: The answer to (a) is “no”. The answer to (b) is “yes”.

Here is an inappropriate ‘problem’ based on the data in this problem: *Which better represents the salaries of employees at this company: the mean or the median?* Outside of a context, this question cannot be answered. If the purpose is to calculate the annual amount allocated to salaries or some related task, then the mean is better. On the other hand, a prospective employee might be more interested in the median or even the first quartile. *Here and in general, the choice of which statistic or statistics to use to best describe given data depends on context, that is, on how the information will be used.*

5: Sally uses a measuring tape to measure the heights of all the children in her class, rounding each measurement to the nearest whole number of inches. Her data are

$$50\,,\,50\,,\,50\,,\,52\,,\,53\,,\,53\,,\,57\,,\,58\,,\,58\,,\,59\,,\,60\,,\,61\,,\,61\,.$$

Find the mean, median, first quartile, and all modes of this collection of data.

Discussion: Notice that there is a unique mode—namely 50. Sometimes modes are mistakenly described as indicators of 'centers of data'. This problem shows that any such phrase is misleading as a description of modes. In fact, modes are far less important than means and medians; modes are very sensitive to small changes in data whereas means and medians are not.

Inherent in data organization is the issue of rounding numbers. It is a good exercise for students who have had algebra to prove that if Sally had re-measured to the nearest half-inch, then—whatever the actual heights of the 13 students—the means of Sally's first data and her second would differ by no more than 1/2, and that the same is true for the medians and quartiles.

Good representations of data accompanied by calculations of important statistics seem to invite one to draw conclusions or make predictions about future events or even to make certain decisions. But in doing so it is easy to make subtle but important errors without having a hint that errors are being made. Throughout their lives, students will be confronted by others making claims on the basis of statistics, and it is thus important for them to learn the variety of subtle but deceiving mistakes that can arise. The problems below, which require the student to look critically at predictions and other uses of data, help promote this learning.

6: What is wrong with the following problem and proposed solution?

> The average snowfall in Minneapolis, Minnesota during the fall-winter-spring snow season is 56.3 inches. Suppose that at the end of February in some particular snow season, the total snowfall thus far is 15.2 inches. What is a good estimate of the amount of snow yet to come during that snow season?
>
> Proposed solution: 56.3-15.2 = 41.1 inches

Discussion: The major error: a possibility has been overlooked—that the 15.2 inches of snow thus far observed might indicate that the total snowfall for the entire snow season will be far below the average of 56.3 inches. (For instance, winter weather patterns are affected by global ocean currents.) We have no information about the average snowfall in years for which the snowfall through February is only 15.2 inches. Indeed, if the snowfall through the entire snow season less one day were 25 inches, would the best estimate for the last day be a 31.3 inch blizzard?!

7: What is wrong with the following problem and proposed solution?

The average snowfall in Minneapolis, Minnesota during the fall-winter-spring snow season is 56.3 inches, and the average for the last three months—March, April, and May—of the snow season is 13.7 inches. Suppose that at the end of February in some particular snow season, the total snowfall so far is 15.2 inches. What is a good estimate for the amount of snow yet to come during that snow season?

Proposed solution: 13.7 inches, the mean for the remaining three months

Discussion: Error: In the proposed solution, the other information in the problem has been ignored, and this should only be done if the snowfall in the last three months is independent of the snowfall during the earlier part of the snow season. The problem gives no definite information about dependence or independence.

Information from previous years would be useful in assessing the question of dependence.

8: An advertisement says that, based on polling by an independent agency, 4 out of 5 dentists prefer Glistening White toothpaste to two other national brands. How might it be that the statement is factually correct but very misleading?

Discussion: There are many ways, so it might be a challenge to formulate a fair rubric for grading.

One way: Maybe the dentists were asked to rank eight different toothpastes and Glistening White was among the three worst but managed to be favored over two other national brands by 4 out of 5 dentists.

Second way: Maybe only 5 dentists were interviewed and the results are fully representative of the views of those 5 dentists. Of course, the person who hears this advertisement tends to imagine 500 dentists out of which 400 were enthusiastic about Glistening White.

Third way: Maybe several different independent agencies ran their own separate polls, say in different parts of the country, with widely varying results. And the Glistening White Company has chosen to only mention the one that is favorable to Glistening White toothpaste.

9: What is wrong with the following question?

The students in Mr. Everson's biology classes are asked to vote for a biological symbol for the school year. The vote is among the following three symbols which have been previously nominated: frog, lizard, and gnu. Here are results of the vote:

	Class 1	Class 2	Class 3
frog	23	2	5
lizard	7	6	16
gnu	4	26	2

What symbol should Mr. Everson choose and why?

Discussion: The appropriate question is: What mistake has Mr. Everson already made? Answer: He did not decide on the method of evaluating the vote and announce that decision before the vote was taken.

This issue is very closely related to an important issue in experimental design. Students should be taught that the plan for any experiment must include the questions to be addressed, a description of the data needed, how that data is to be collected, and how the data is to be analyzed, and this plan should be in place before the data is obtained. This should be included in the science curriculum where it can then be implemented in the science classroom as an illustration of the scientific method.

10: To carry out an experiment, a science class measures a quantity Q at 3 one-second intervals. If at time $t = 0$, the quantity is given by $Q = 5$, at time $t = 1$ the quantity is $Q = 10$, and at time $t = 2$ the quantity is $Q = 15$, can you predict the quantity at time $t = 3$?

Discussion: Answer: there is no basis for making this prediction without additional information. Certainly, the quantity may be growing linearly, so that at $t = 3$ one would expect $Q = 20$, but the quantity may be better modeled by the equation $Q(t) = 5(t-1)^3 + 10$, or by $Q(t) = 10 - 5\cos(\pi t/2)$, or by infinitely many others.

Note that the above problems do not require advanced high-school mathematics. By contrast, algebra is useful for writing the formulas of statistics, as well as for understanding how changes of scale and accuracy of measurement affect statistical descriptions of data. Algebra, and in many cases Calculus, is needed for accurate modeling and predicting.

Probability

The relation of the mathematical subject of probability to the real world is similar to the relation of high-school geometry to the real world. No real-world triangles are ever exactly equilateral; similarly, no coins are ever exactly equally likely to come up heads as to come up tails. In geometry we do, nevertheless, spend considerable time talking about equilateral triangles and successfully apply the knowledge to the real world. And in probability we do discuss flipping fair coins and we apply the knowledge to many real-world coins as well as to situations where no coins are involved. (A fair coin is one that shows heads with probability 1/2 and tails with probability 1/2.)

Before giving problems on probability, we first consider one matter of terminology. Consider the experiment of flipping a coin—an actual physical coin. One might be interested in the probability that it comes up heads. To estimate this probability one might flip the coin a large number, say 1000, times. Alternatively, one might observe that the coin is almost symmetrical and therefore conclude that the probability is close to 1/2. These are two methods of obtaining an estimate; it is misleading to call the result of one method 'experimental probability' and the other result 'theoretical probability'. It would be useful if state standards were not to employ these two terms. Rather, there is probability and the various methods of estimating probability.

The proper term for describing the proportion of times some particular event happens when the same experiment is repeated many times is ‘relative frequency’.

11: Suppose that a fair coin is flipped three times and that the flips are independent. Calculate the probability that exactly two heads are obtained.

Discussion: One thing that can help simplify the preceding problem is to explicitly write down the sample space. We are pleased that many state standards introduce sample spaces in middle school.

12: There are 2 red markers in a bag and 3 green markers, which are otherwise indistinguishable. Lillian draws two markers at random, one after the other, leaving 3 markers in the bag. What is the probability that Lillian draws both red markers? What is the probability that she draws a red marker on her first draw or a green marker on her second draw? What is the probability that she draws a red marker on her first draw and a green marker on her second draw?

Discussion: The numbers in the preceding problem are small enough for one to write down an explicit sample space. The questions are about subsets of that sample space and some of the questions involve intersections and unions of some easily identifiable subsets. Handling the language of sets in this manner along with related words such as ‘or’ and ‘and’ uses an important mathematical skill. As this question and discussion illustrate, some aspects of probability reinforce important aspects of other areas of mathematics.

13: There are $r \geq 2$ red markers in a bag and $g \geq 1$ green markers, which are otherwise indistinguishable. Lillian draws two markers at random, one after the other, leaving $r + g - 2$ markers in the bag. What is the probability that Lillian draws two red markers? What is the probability that she draws a red marker on her first draw or a green marker on her second draw? What is the probability that she draws a red marker on her first draw and a green marker on her second draw?

Discussion: This problem is a higher-level version of the preceding problem. It involves algebra, and requires the student who is taking a sample-space point of view to imagine the details of the sample space rather than writing them all down.

14: For the setting of Problem 12, represent the sample space as a tree. Then use the tree and the probabilities associated with its branches to calculate the probability that Lillian’s first draw was green, given that her second draw was red.

Discussion: Conditional probability, which shows how to use partial information in a systematic manner, and calculations based on trees do belong in high school standards. They have the feature of highlighting the subtlety of the subject, and therefore can help promote critical reasoning about data analysis.

15: A furniture manufacturer produces 20 chairs to identical specifications, but it happens that two of them have subtle defects. Three chairs are chosen at random. What is the probability that none of the three are defective? Also, what is the expected number of defectives in this experiment?

Discussion: Means, medians, standard deviations, and variances are calculated for probability distributions as well as for collections of numerical data. This use of the same terms in two different settings can create confusion. With respect to this double usage, the synonyms for 'mean' can play a useful role. One uses 'average' for collections of data but never for probability distributions. The terms 'expectation', 'expected value', and 'expected number' are synonyms for 'mean' in the context of probability distributions but not for collections of data.

16: Suppose that 75% of the workers in a certain company drive to their job. If 3 workers selected at random are asked to stay late to help move furniture, is it more likely or less likely that all 3 have driven to work?

Discussion: This is a problem that every student should be able to solve. The answer is less likely: the probability is 27/64, which is less than 1/2.

In writing problems such as this, one must be careful concerning the issue of independence. For example, if the problem were changed slightly to read *"Suppose that 75% of the workers in a certain company drive to their job. If 3 workers meet for coffee at break, is it more likely or less likely that all 3 have driven to work?"*, then the problem is flawed. There is no reason to think that the workers meeting over coffee are a random sample; for instance, maybe people who carpool are more likely to have coffee together.

5. Lengthy Calculations

Many real-world problems in mathematics are computationally difficult. However, the problems actually presented to students are usually restricted in type or carefully contrived to be easily worked. Our hope as teachers is that if students can do the simple problems then they can do longer problems that use the same methods or ideas. This problem set focuses on such longer problems. Though they may be too long for a test, it would be valuable to include a few explicit "long problems" in the curriculum. For instance, when learning multiplication of multi-digit integers most practice problems have 2-3 digits. To this could be added a few multiplications of two 4-digit numbers. We suggest that these be explicitly labeled "long problems" and not scattered in with the others. Further they should be presented only after similar short problems have been well mastered so there is no qualitative reason for students to have trouble with them. If short-problem skills do generalize then students should get significant pride and satisfaction from discovering this.

Let us remark that some state standards give the impression that problems become appropriate for higher grade levels as the number of digits increases. To us a more important indicator of difficulty is structure. For instance, addition of fractions is easiest to understand and perform when all the denominators are the same, somewhat more difficult when the denominators are not the same but one of the denominators is also the least common denominator, and still more difficult when neither of the preceding conditions holds. When long problems do not involve new elements of structure, we believe that students who have mastered short problems should be encouraged to do long problems on the same topic. Thus we believe that students who obtain a good command of the multiplication of two multi-digit numbers in a given school year, for example, should be encouraged to do long problems on this topic without waiting for an additional year or more to go by. The same principle applies at all levels of K-12 education, including high school algebra and beyond.

The problems in this section are meant to be solved, once again, with pencil-and-paper. Admittedly, at a later stage of development some of these long problems might be turned over to a calculator, but we believe that there is genuine value in being able to do such problems by hand.

For the more elementary problems, we do not write explicit examples but simply describe the kind of calculation to be done.

1: Add two 10-digit numbers. Add a 10-digit number and an 8-digit number. Add four 5-digit numbers.

Discussion: At the lower grades, some students might find their lack of knowledge of the names of large numbers a psychological roadblock. After solving such a problem, students should realize that they can do such a calculation without the names. Of course, it is also important that students learn what the names of 10-digit numbers are.

2: Subtract one 10-digit number from another. Subtract an 8-digit number from a 10-digit number.

3: Multiply two 4-digit numbers. Multiply two 9 or 10-digit numbers ending in strings of 5 or 6 zeroes (e.g. $4,201,000,000 \times 176,000,000$). Multiply two 9 or 10-digit numbers ending in strings with 4 or 5 zeroes and one non-zero digit (e.g. $4,201,000,007 \times 176,000,060$)

4: Divide a 7-digit number by a 3-digit number. Divide a 10-digit number ending in 5 zeroes by a 5-digit number ending in 3 zeroes. Give the answers as quotients with remainders.

5: Without using scientific notation, calculate

$$0.0000000067 \times 5000000\,.$$

Then do the calculation again by first writing each of given numbers in scientific notation.

6: Write $5/17$ as a repeating decimal.

Discussion: When the students do the long division, they can see why two 1's in succession do not necessarily indicate that the repeating pattern has started. Also note that the period is 16 and that many calculators do not display this many digits.

7: Write $.378\overline{2474}$ as a fraction.

8: Add a stack of decimal numbers, a short stack of fractions and a short stack of mixed numbers.

The above problems concern computations with numbers. We next give some problems in the area of algebra in order to emphasize that, as noted above, fluency in algebraic computations also ought to include working several lengthy problems. Note that problems 10 and 11 involve geometry as well as algebra. Problems 11–14 below may be more suitable for an Algebra 2 course.

9: Write both

$$(x+y)^6 \qquad \text{and} \qquad (u-2v)^6$$

as sums of monomials. Then check the formulas by inserting $x = y = 1$ and $u = v = 1$. (Note: this is only a partial check of the formulas. Another, arguably better, check is $x = u = 10$, $y = v = 1$.)

10: Find where the line meets the circle:

$$y = 3x \qquad \text{and} \qquad x^2 + y^2 = 36.$$

11: Solve the 3 simultaneous equations:

$$x - y + z = 0 \qquad y - 2z = 0 \qquad x^2 + y^2 + z^2 = 16.$$

Discussion: This problem is algebraic but it answers a geometric problem as well. The first two equations are planes, and they intersect in a line. Solving the 3 equations simultaneously gives the points that are on the intersection of this line with the sphere $x^2 + y^2 + z^2 = 16$.

12: Write $(a + b + 2c - d)^3$ as a sum of monomials.

Discussion: One way to do this is to apply the standard formula $(x+y)^3 = x^3+3x^2y+3xy^2+y^3$ with $x = a + b + 2c$ and $y = -d$. The computation then reduces to several simpler problems.

13: Solve for w, x, y, z:

$$\begin{array}{rrrrcr} w & +x & +y & +z & = & 10 \\ w & & +3y & -2z & = & 2 \\ -3w & +x & -9y & +7z & = & 0 \\ -2w & -3x & -6y & -7z & = & -18. \end{array}$$

Discussion: The method of choice here is the Gaussian Elimination process and not Cramer's Rule.

14: Find the value of the function $V(x, y, z) = 8xyz$ if $x, y, z,$ are positive numbers and they satisfy the equations

$$\begin{aligned} x^2 + \frac{y^2}{4} + \frac{z^2}{9} &= 1 \\ 8yz + 2\lambda x &= 0 \\ 8zx + 2\lambda \frac{y}{4} &= 0 \\ 8xy + 2\lambda \frac{z}{9} &= 0 \end{aligned}$$

for some quantity λ.

Discussion: This calculation arises in calculus [1]. When students encounter such a problem in college, the discussion is on obtaining the above equations. Once they do so, it is assumed that students have the algebra skills to solve them. Indeed, this is regarded as the easy step; if such a step actually causes students to struggle then it distracts them from the central calculus concepts involved.

[1] in finding the box of largest volume with each edge parallel to one of the coordinate axes that is contained in the ellipsoid $x^2 + y^2/4 + z^2/9 = 1$